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ECE 204 *Numerical methods*

The inner product and orthogonal bases

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Introduction

- In this topic, we will
  - Reviewed the dot product and introduced a new notation
  - Described orthogonality, the angle between vectors, and defined the 2-norm
  - Described the projection of a vector **v** onto a vector **u**
    - The best approximation of **v** as a scalar multiple of **u**
  - Described how to approximate a vector **v** as a linear combination of orthogonal set of vectors
  - Review the useful properties of the inner product

The inner product

- Recall that the dot product between two vectors is defined as

$$\mathbf{u} \bullet \mathbf{v} = \sum_{k=0}^{n} u_k^* v_k$$

$$\mathbf{u} + \mathbf{v}$$
$$\alpha \mathbf{u}$$
$$\mathbf{u} \times \mathbf{v}$$

- Let's define a more general notation: the *inner product*:

$$\langle \mathbf{u}, \mathbf{v} \rangle = \sum_{k=0}^{n} u_k^* v_k$$

  – If we are dealing with real vectors, this simplifies to

$$\langle \mathbf{u}, \mathbf{v} \rangle = \sum_{k=0}^{n} u_k v_k$$

The inner product

- Applications of the inner product include being able to determine when two vectors are *orthogonal* or at right angles
  - Two vectors **u** and **v** are orthogonal if

$$\langle \mathbf{u}, \mathbf{v} \rangle = 0$$

- We can define an *angle* between two vectors as

$$\theta = \cos^{-1}\left(\frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\|\mathbf{u}\|_2 \, \|\mathbf{v}\|_2} \right)$$

- We can also define a *distance* between two vectors:

$$\|\mathbf{u}\|_2 = \sqrt{\langle \mathbf{u}, \mathbf{u} \rangle} = \sqrt{\sum_{k=0}^{n} |u_k|^2}$$

The projection

- The projection of **v** onto **u** is the best approximation of **v** by a scalar multiple $a\mathbf{u}$
  - That is, $a\mathbf{u}$ is that scalar multiple of **u** that minimizes:

$$\left\|\mathbf{v} - a\mathbf{u}\right\|_2$$

  - That scalar is found by the projection:

$$a\mathbf{u} = \operatorname{proj}_{\mathbf{u}}(\mathbf{v}) = \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\langle \mathbf{u}, \mathbf{u} \rangle}\mathbf{u}$$

  - Thus, $a = \dfrac{\langle \mathbf{u}, \mathbf{v} \rangle}{\langle \mathbf{u}, \mathbf{u} \rangle}$

Orthogonal basis

- If $\mathbf{u}_1, \ldots, \mathbf{u}_n$ forms an orthogonal basis for $\mathbf{R}^n$ or $\mathbf{C}^n$, then given $\mathbf{v}$, we define

$$a_k = \frac{\langle \mathbf{u}_k, \mathbf{v} \rangle}{\langle \mathbf{u}_k, \mathbf{u}_k \rangle}$$

- Then

$$\mathbf{v} = a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + \cdots + a_n\mathbf{u}_n = \sum_{k=1}^{n} \mathrm{proj}_{\mathbf{u}_k}(\mathbf{v}) = \sum_{k=1}^{n} a_k\mathbf{u}_k$$

  - If some of the coefficients are sufficiently small, discarding those terms still gives a reasonable approximation

Best approximation

- If $\mathbf{u}_1, \ldots, \mathbf{u}_m$ forms an orthogonal set in $\mathbf{R}^n$ or $\mathbf{C}^n$, then given $\mathbf{v}$, we define

$$a_k = \frac{\langle \mathbf{u}_k, \mathbf{v} \rangle}{\langle \mathbf{u}_k, \mathbf{u}_k \rangle}$$

- Then

$$\mathbf{v} \approx a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + \cdots + a_m\mathbf{u}_m = \sum_{k=1}^{m} \operatorname{proj}_{\mathbf{u}_k}(\mathbf{v}) = \sum_{k=1}^{m} a_k\mathbf{u}_k$$

Example

- For example, $\mathbf{v} = \begin{pmatrix} 2.8 \\ -3.7 \\ 1.5 \\ -0.9 \end{pmatrix} = 2.8\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} - 3.7\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + 1.5\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} - 0.9\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

$$\mathbf{v} = -0.075\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} - 0.375\begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix} + 2.225\begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} + 1.025\begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}$$

$$-0.375\begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix} + 2.225\begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} + 1.025\begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2.875 \\ -3.625 \\ 1.575 \\ -0.825 \end{pmatrix}$$

Example

- Recall the following properties of the inner product:
  - First, $\langle \mathbf{u}, \mathbf{u} \rangle \geq 0$ and $\langle \mathbf{u}, \mathbf{u} \rangle = 0$ if and only if $\mathbf{u} = \mathbf{0}$
  - Second, $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle^*$
  - Third, $\langle \mathbf{u}, \alpha \mathbf{v} \rangle = \alpha \langle \mathbf{u}, \mathbf{v} \rangle$ and $\langle \mathbf{u}, \mathbf{v} + \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{u}, \mathbf{w} \rangle$

- All the useful properties of the inner product are the consequence of these three properties

Summary

- Following this topic, you now
  - Know the dot product and our new angle-bracket notation for the inner product
  - Have reviewed orthogonality, the angle between vectors, and definition of the 2-norm
  - Understand how to find the best approximation of **v** by a scalar multiple of a given vector **u**: the projection of **v** onto **u**
  - Know how to approximate a vector **v** as a linear combination of orthogonal vectors
  - Have seen the useful properties of the inner product

References

[1] https://en.wikipedia.org/wiki/Inner_product

[2] https://en.wikipedia.org/wiki/Projection_(linear_algebra)

[3] https://en.wikipedia.org/wiki/Norm_(mathematics)

[4] https://en.wikipedia.org/wiki/Orthogonality

Acknowledgments

None so far.

Colophon

These slides were prepared using the Cambria typeface. Mathematical equations use Times New Roman, and source code is presented using `Consolas`. Mathematical equations are prepared in MathType by Design Science, Inc.

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The photographs of flowers and a monarch butter appearing on the title slide and accenting the top of each other slide were taken at the Royal Botanical Gardens in October of 2017 by Douglas Wilhelm Harder. Please see

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